Math 2010 Week 3

Polar coordinates in $\mathbb{R}^2$

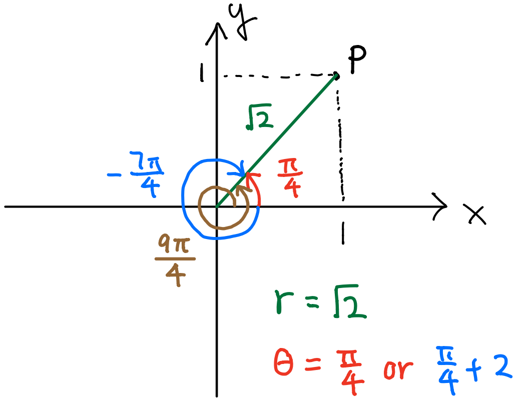
(Au 1.5.2
Thomas 11.3, 11.4)

A point $P=(x,y)\in\mathbb{R}^2$ can be represented by

$r=\sqrt{x^2+y^2}$ = distance from origin

θ = angle from the positive x-axis to $\overrightarrow{OP}$ in counter-clockwise direction

eg $P=(1,1)$



$r=\sqrt{2}$

$\theta=\frac{\pi}{4}$ or $\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

Rmk

① For $P=(0,0)$ $\begin{cases} r=0 \\ \theta \text{ is not (uniquely) defined.} \end{cases}$

② Different conventions for ranges of r and θ

$r\in[0,\infty)$ or $r\in\mathbb{R}$ ← our textbook

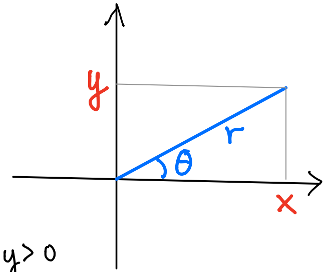
$\theta\in[0,2\pi)$ or $\theta\in\mathbb{R}$

In this course, we usually take

$r\in[0,\infty)$ and $\theta\in\mathbb{R}$

Change of coordinates formula

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$



$r=\sqrt{x^2+y^2}$

$\theta=\tan^{-1}\left(\frac{y}{x}\right)$ for $x,y>0$

Similar formula for θ in other quadrants

Curves in Polar Coordinates

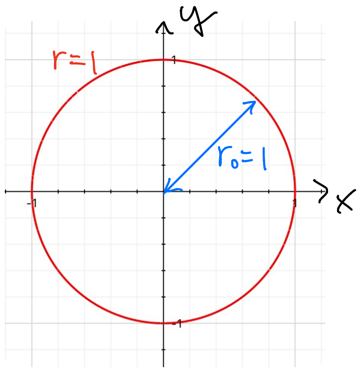
eg Circle with radius $r_0 > 0$, centered at origin

Polar Equation

$$r = r_0$$

Parametric form

$$\begin{cases} r = r_0 \\ \theta = t, \quad t \in [0, 2\pi] \end{cases}$$



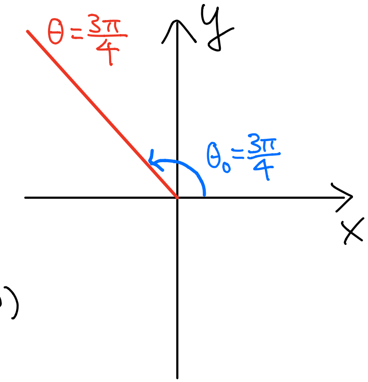
eg Half ray from origin

Polar Equation

$$\theta = \theta_0$$

Parametric form

$$\begin{cases} r = t, \quad t \in [0, \infty) \\ \theta = \theta_0 \end{cases}$$



eg Archimedes Spiral

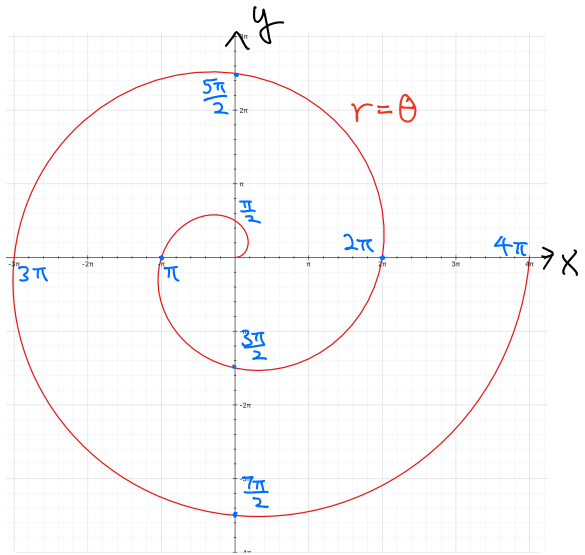
Let $k > 0$ be a constant

Polar Equation

$$r = k\theta$$

Parametric form

$$\begin{cases} r = kt \\ \theta = t \end{cases}, \quad t \in [0, \infty)$$



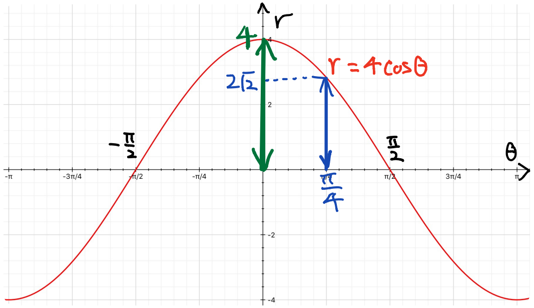
Picture for $t \in [0, 4\pi]$

eg $r = 4\cos\theta$

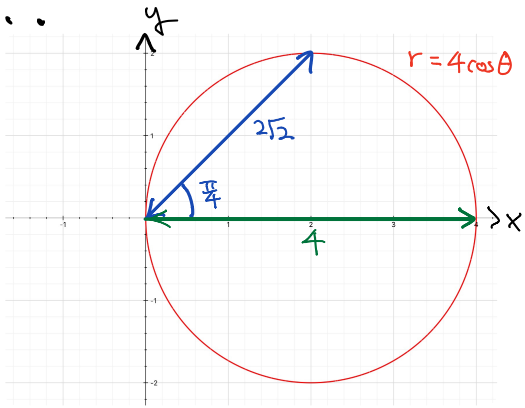
Our convention: $r \geq 0 \Rightarrow \cos\theta \geq 0$

$\therefore$ θ is in quadrant I or IV

Take $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\therefore$



Observation: It looks like a circle... yes!

Note $r = 4\cos\theta \Rightarrow r^2 = 4r\cos\theta$

$$\Rightarrow x^2 + y^2 = 4x$$

$$\Rightarrow (x-2)^2 + y^2 = 2^2$$ (Circle with radius = 2 centered at (2, 0))

eg $r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}$

Try → $\theta = 0 \Rightarrow r = 2$

$\theta = \frac{\pi}{2} \Rightarrow r = 2$

$\theta = \frac{\pi}{4} \Rightarrow r = \sqrt{2}$

Note $\sqrt{2} = r\cos(\theta - \frac{\pi}{4})$

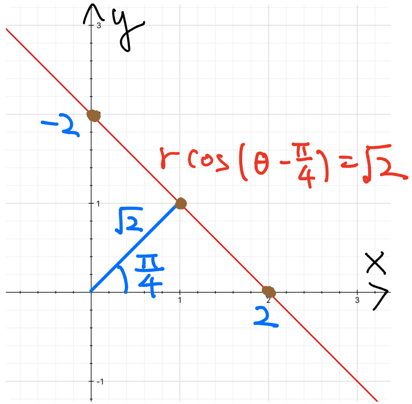
$$= r\left(\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}\right)$$

$$= \frac{r}{\sqrt{2}}\cos\theta + \frac{r}{\sqrt{2}}\sin\theta$$

$$= \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

$\therefore$ $x + y = 2$

i.e. a line



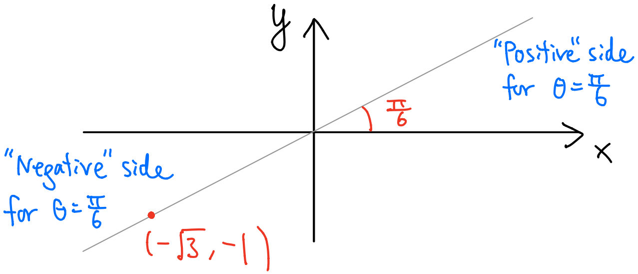
$\boxed{\underbrace{r \in [0,\infty)}_{\text{Our convention}} \quad \text{vs} \quad r \in (-\infty, \infty)}$

It is sometimes convenient to allow $r<0$ and interpret

$$(x,y) = (r\cos\theta, r\sin\theta)$$
$$= \left(-|r|\cos\theta, -|r|\sin\theta\right)$$
$$= -\left(|r|\cos\theta, |r|\sin\theta\right)$$

eg $r=-2$, $\theta = \frac{\pi}{6}$

$$\Rightarrow (x,y) = \left(-2\cos\tfrac{\pi}{6}, -2\sin\tfrac{\pi}{6}\right) = (-\sqrt{3}, -1)$$

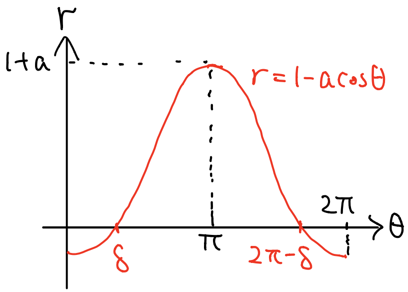


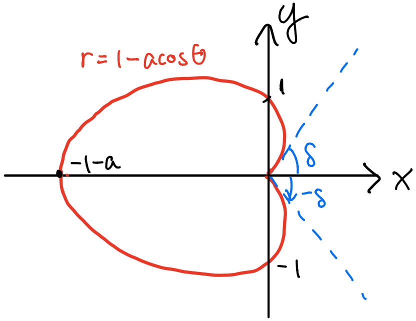
eg $r = 1 - a\cos\theta$, where $a>1$ is a constant.

Case 1 If we require $r \geq 0$, then

$$1 - a\cos\theta \geq 0 \Rightarrow \cos\theta \leq \frac{1}{a} < 1$$

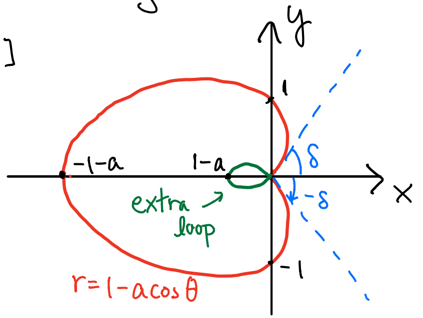
$\therefore$ θ cannot run through the whole interval $[0, 2\pi]$ but only $[\delta, 2\pi-\delta]$, where $\delta = \cos^{-1}\frac{1}{a}$





Case 2 If we allow $r \in \mathbb{R}$ to be negative

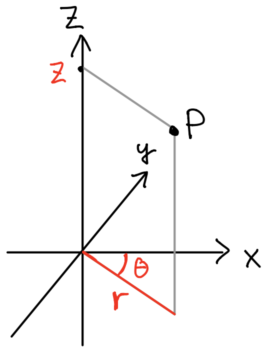
θ can run through $[0, 2\pi]$ and we get a smooth, self-intersecting curve



Some Coordinates Systems in $\mathbb{R}^3$

Cylindrical Coordinates

$$(x,y,z) \xrightarrow{\text{Express } x,y \text{ using polar}} (r,\theta,z)$$



Formulas

$x = r\cos\theta$

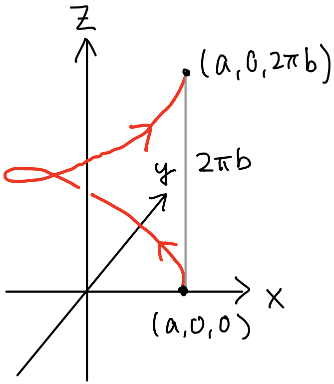
$y = r\sin\theta$

$z = z$

eg (Helix)

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases}$$

$t \in [0, 2\pi]$

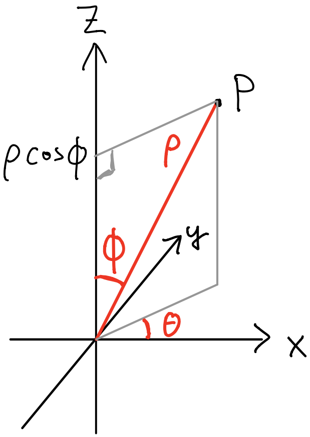


Spherical Coordinates

Describe $P \in \mathbb{R}^3$ by

$$\begin{cases} \rho = \text{distance from origin} \\ \quad = \sqrt{x^2+y^2+z^2} \\ \theta = \theta \text{ as in cylindrical coordinates} \\ \phi = \text{angle from positive } z\text{-axis to } \overrightarrow{OP} \end{cases}$$

<u>Rmk</u> $\phi \in [0, \pi]$



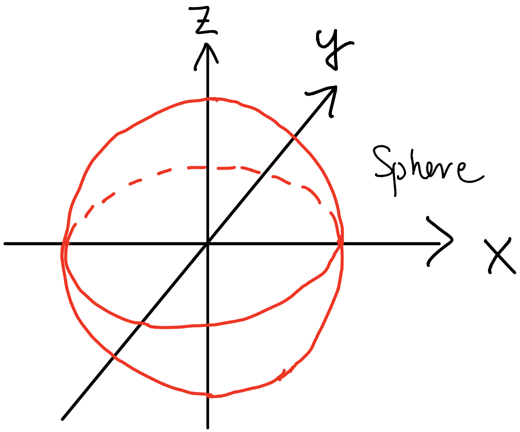
Formulas

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

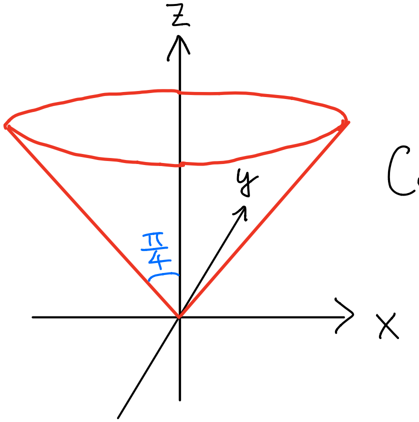
$z = \rho\cos\phi$

eg $\rho = 2$



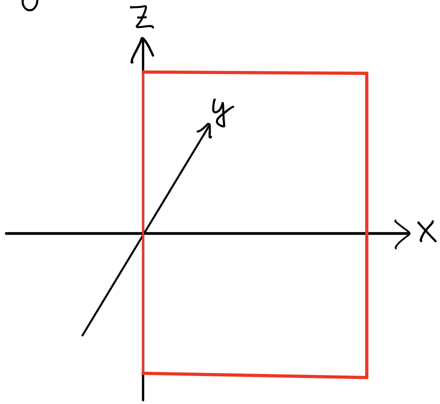
Sphere

eg $\phi = \frac{\pi}{4}$



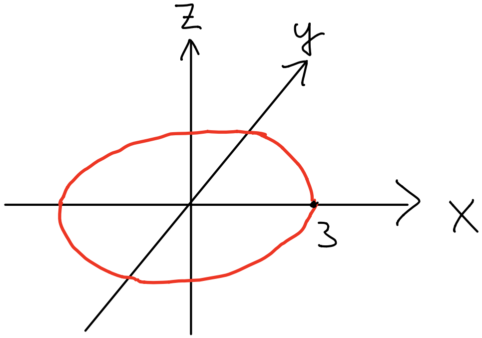
Cone

eg $\theta = 0$



Half-plane

eg



Circle

<u>Equations</u>

$$\begin{cases} \rho = 3 \\ \phi = \frac{\pi}{2} \end{cases}$$

or

<u>Parametric form</u>

$$\begin{cases} \rho = 3 \\ \theta = t \quad , \ t \in [0, 2\pi] \\ \phi = \frac{\pi}{2} \end{cases}$$

Topological Terminology in $\mathbb{R}^n$

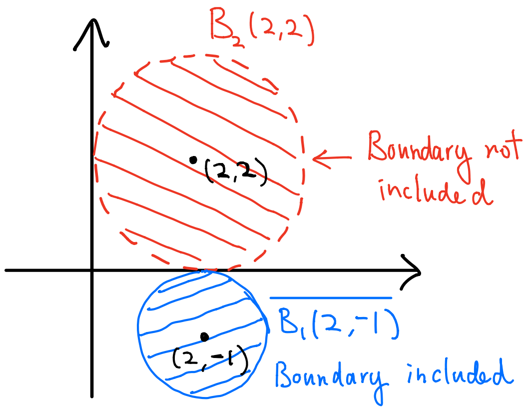
Let $\vec{x}_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Define

$$B_\varepsilon(x_0) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < \varepsilon\}$$

= <u>open ball</u> with radius ε and centered at $\vec{x}_0$.

$$\overline{B_\varepsilon(x_0)} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \varepsilon\}$$

= <u>closed ball</u> with radius ε and centered at $\vec{x}_0$.



<u>Defn</u> Let $S \subseteq \mathbb{R}^n$. Define the following sets:

① The <u>interior of S</u> is the set

$$\text{Int}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(x) \subset S \text{ for some } \varepsilon > 0\}$$

Points in Int(S) are called <u>interior points</u> of S.

② The <u>exterior of S</u> is the set

$$\text{Ext}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(x) \subset \mathbb{R}^n \setminus S \text{ for some } \varepsilon > 0\}$$

Points in Ext(S) are called <u>exterior points</u> of S.

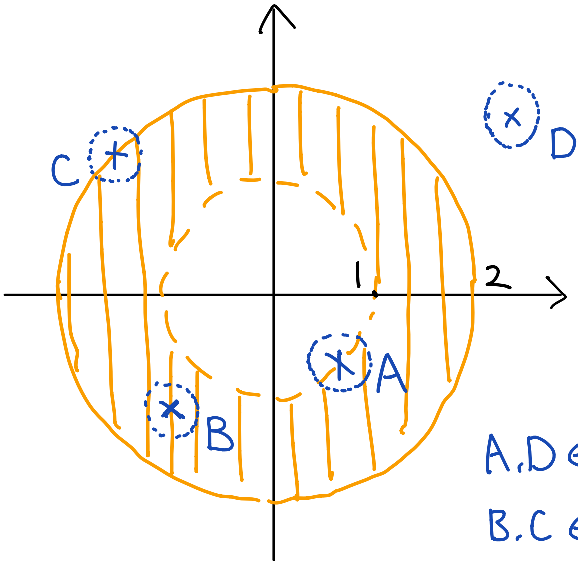
③ The <u>boundary of S</u> is the set

$$\partial S = \left\{ \vec{x} \in \mathbb{R}^n : \begin{matrix} B_\varepsilon(x) \cap S \neq \phi \\ B_\varepsilon(x) \cap \mathbb{R}^n \setminus S \neq \phi \end{matrix} \text{ for any } \varepsilon > 0 \right\}$$

Points in ∂S are called <u>boundary points</u> of S.

eg

$$S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 \leq 4\} \subseteq \mathbb{R}^2$$



$A, D \notin S$

$B, C \in S$

A, C are boundary points of S

B is an interior point of S

D is an exterior point of S

Prop Let $S \subseteq \mathbb{R}^n$. Then

① $\mathbb{R}^n$ is the disjoint union of $\text{Int}(S)$, $\text{Ext}(S)$ and ∂S.

② $\text{Int}(S) \subseteq S$, $\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$

a point in ∂S may or may not be in S

(See A, C in last example)

Defn A subset $S \subseteq \mathbb{R}^n$ is called

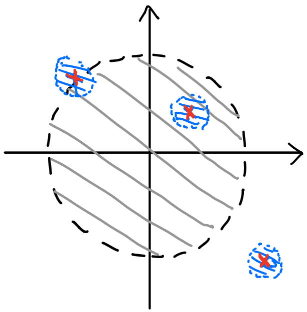
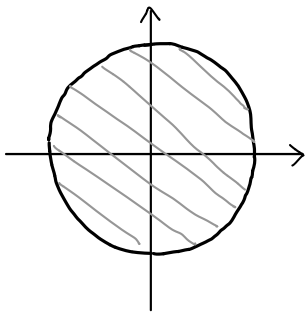
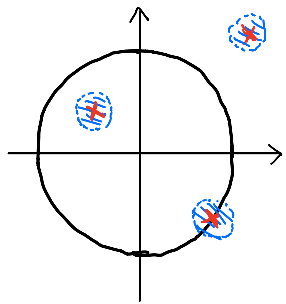
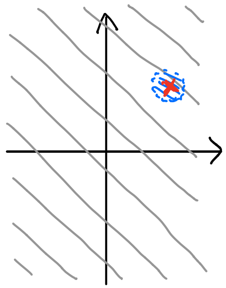
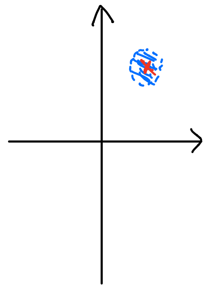
① open if $\forall x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$

② closed if $\mathbb{R}^n \setminus S$ is open

Equivalent definition A subset $S \subseteq \mathbb{R}^n$ is

① open if $S = \text{Int}(S)$

② closed if $S = \text{Int}(S) \cup \partial S$

Subset $S \subseteq \mathbb{R}^2$	$B_1(0,0) = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$	$\overline{B_1(0,0)} = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$	$S^1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$	$\mathbb{R}^2$	$\emptyset$
$\operatorname{Int}(S)$	$B_1(0,0)$	$B_1(0,0)$	$\emptyset$	$\mathbb{R}^2$	$\emptyset$
$\operatorname{Ext}(S)$	$\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\} = \mathbb{R}^2 \setminus \overline{B_1(0,0)}$	$\mathbb{R}^2 \setminus \overline{B_1(0,0)}$	$\mathbb{R}^2 \setminus S^1$	$\emptyset$	$\mathbb{R}^2$
∂S	S^1	S^1	S^1	$\emptyset$	$\emptyset$
Open?	✓	✗	✗	✓	✓
Closed?	✗	✓	✓	✓	✓
Picture					

Rmk

① There are exactly two subsets of $\mathbb{R}^n$ which are both open and closed

$$\mathbb{R}^n \quad \text{and} \quad \phi$$

② Some subsets of $\mathbb{R}^n$ are neither open or closed

eg. $\{(x,y)\in\mathbb{R}^2 : 1 < x^2+y^2 \le 4\} \subseteq \mathbb{R}^2$

$(0,1] \subseteq \mathbb{R} \qquad \mathbb{Q} \subseteq \mathbb{R}$

③ For any $S \subseteq \mathbb{R}^n$

$\mathrm{Int}(S), \mathrm{Ext}(S)$ are open in $\mathbb{R}^n$

∂S is closed in $\mathbb{R}^n$

Two more definitions

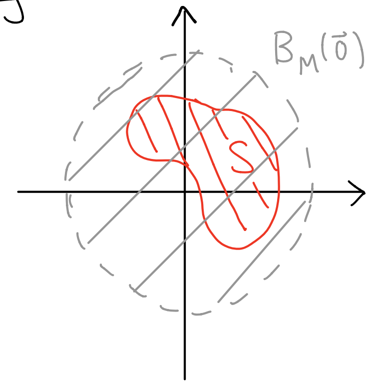
Let $S \subseteq \mathbb{R}^n$ be a subset

① S is called <u>bounded</u> if $\exists M > 0$ such that

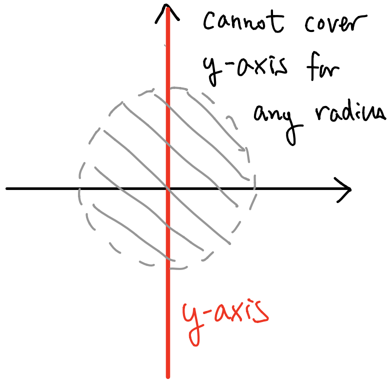
$$S \subseteq B_M(\vec{0}) = \{\vec{x} \in \mathbb{R}^n : \|x\| < M\}$$

S is called <u>unbounded</u> if S is not bounded

eg



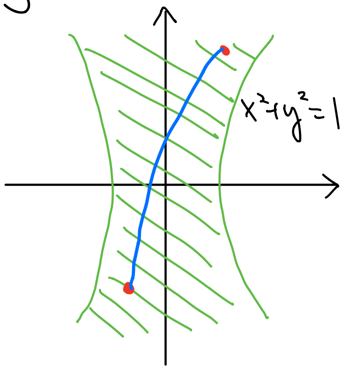
S is bounded



y-axis is unbounded

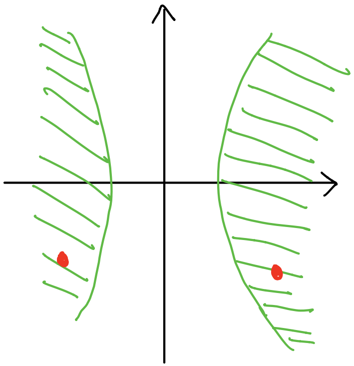
② S is called <u>path-connected</u> if any two points in S can be connected by a curve in S

<u>eg</u>



$\{(x,y) : x^2-y^2 \le 1\}$

path-connected



$\{(x,y) : x^2-y^2 \ge 1\}$

Not path-connected

<u>Rmk</u> There is also a different notion called connectedness. However, we will not discuss it.

<u>Jordan Curve Theorem</u>

A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two path-connected components, with one bounded and one unbounded.

<u>eg</u>